【Article】

Investigation of Prawitz's completeness conjecture in phase semantic framework

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Abstract

In contrast to the usual Tarskian set-theoretic semantics, in which the notion of validity is defined for sentences, in proof-theoretic semantics of Prawitz and Schroeder-Heister, the validity is defined for proofs or derivations. By defining the validity for a broader class of inferences including invalid inferences, Prawitz (1973) conjectured completeness of proof-theoretic semantics. In this article, we investigate the Prawitz's completeness conjecture by giving a characterization of proof-theoretic semantics in the phase semantic framework of Okada & Takemura (2007).

1 Introduction

In traditional Tarskian set-theoretical semantics, the notion of validity is defined for sentences or formulas in terms of truth conditions thereof. For example, a sentence $A \wedge B$ (the conjunction of sentences A and B) is valid if and only if A is valid and B is valid. In contrast, in prooftheoretic semantics of Prawitz and Schroeder-Heister, the notion of validity is firstly defined for derivations or proofs in terms of constructions thereof. For example, a derivation t of $A \wedge B$ is valid if and only if t reduces (or is equivalent) to a pair of derivations $\langle u, v \rangle$ such that u of A is valid and v of B is valid. Then, a sentence is valid if there exists a valid derivation thereof.

Proof-theoretic semantics goes back to remarks made by Gentzen, who introduced the system of natural deduction consisting of introduction and elimination inference rules for every logical connective. In (Gentzen 1934), he remarked that an introduction rule in the natural deduction system gives the meaning of the logical connective in question, and this meaning justifies the corresponding elimination rule. By developing this idea, Prawitz (Prawitz 1971) formally defined the proof-theoretic validity for deductions in the system of natural deduction for intuitionistic logic. A deduction is valid, if it is in canonical form that ends with an introduction rule, and if its immediate subdeductions are valid. Thus, each introduction rule is valid in its own right by definition. On the other hand, a deduction ends with an elimination rule is valid if it reduces to the canonical form. Thus, each elimination rule should be justified to be valid based on the meaning defined by the corresponding introduction rule. The term "proof-theoretic semantics" was coined by Schroeder-Heister, and he has extensively analyzed and developed it. See, for example, (Schroeder-Heister 2006; Piecha & Schroeder-Heister 2016; Schroeder-Heister 2012) for recent studies on proof-theoretic semantics.

Prawitz further extended the notion of validity to a broader class of inferences called arguments, which are essentially trees of formulas built from arbitrary inferences including invalid inferences. Prawitz then formulated his completeness conjecture as follows; cf. (Prawitz 1973; Prawitz 2013).

Prawitz's conjecture: *All inference rules that are valid in the sense of proof-theoretic semantics hold as derived rules in Gentzen's system of natural deduction for intuitionistic logic.*

This conjecture is still undecided and continues to be discussed extensively by logicians and philosophers. See (Piecha&Schroeder-Heister 2016; Schroeder-Heister 2006; Schroeder-Heister 2012) for the background, related research, and recent discussions.

Another origin of proof-theoretic semantics is the so-called "computability argument" to prove the proof normalization theorem. In (Prawitz 1971), the proof-theoretic validity is introduced in the appendix after a discussion of the proof normalization theorem of natural deduction.

In (Okada & Takemura 2007), we introduced phase semantic framework to prove normalization via completeness. Phase semantics was introduced by Girard (Girard 1987) as the usual set-theoretical semantics for linear logic. See, for example, (Okada 2002) for phase semantics.

In (Okada & Takemura 2007), phase semantics for intuitionistic logic is extended by augmenting proof-terms, i.e., the usual λ-terms. Then the completeness with respect to such extended phase semantics implies the normal form theorem of intuitionistic logic. Our phase semantic framework can be considered as one of the semantic variations of the computability argument.

We investigate Prawitz's completeness conjecture in our phase semantic framework. In Section 2, we first review natural deduction for our atomic second-order propositional intuitionistic logic $\mathsf{IL}^2_{\to\forall}\mathsf{at}$. Then, we introduce phase semantics for $\mathsf{IL}^2_{\to\forall}\mathsf{at}$, and prove the soundness and the completeness theorems of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ with respect to our phase semantics. In Section 3, we first review Prawitz and Schroeder-Heister's proof-theoretic semantics. Then, we show that Prawitz and Schroeder-Heister's proof-theoretic semantics is characterized as a phase model consisting only of closed proof-terms. Finally, in Section 4, we discuss Prawitz's completeness conjecture from the view point of phase semantics.

2 Phase semantics

We review our system $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ in Section 2.1, and introduce our phase semantics in Section 2.2. Then, we show the soundness and completeness theorems in Section 2.3.

2.1 Syntax of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$

Our *atomic second-order intuitionistic propositional logic* $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ is essentially the same system as $\mathbf{F}_{at}$ introduced by (Ferreira 2006; Ferreira & Ferreira 2013). $\mathbf{F}_{at}$ is a subsystem of Girard's system $\mathbf{F}$ (cf. (Girard, Taylor & Lafont 1989)), where the comprehension, i.e., $\forall$-elimination rule in natural deduction, is applied only to an atomic formula. Under the restriction, basic properties such as normalization are proved in similar ways as first order cases. Moreover, even if we restrict the comprehension, it is shown that intuitionistic connectives $\wedge$, $\vee$, $\bot$, $\exists$ are definable in $\mathbf{F}_{at}$ as follows.

- $A \wedge B := \forall X.((A \to (B \to X)) \to X)$
- $A \vee B := \forall X.((A \to X) \to ((B \to X) \to X))$
- $\bot := \forall X.X$
- $\exists X.A := \forall Y.(\forall X.(A \to Y) \to Y)$

See (Ferreira 2006) for the detailed discussion on the definability of these connectives.

Our system is slightly different from $\mathbf{F}_{at}$ by considering a type assignment system, which includes untypable terms.

Definition 2.1 Formulas of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ are defined as follows.

$$A, B := \quad X \mid A \to B \mid \forall X.A$$

Definition 2.2 Proof-terms, or **terms** for short, of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ are the following λ-terms:

$$t, s := \quad x \mid \lambda x^A.t \mid ts \mid \Lambda X.t \mid tX$$

Note that in application terms of the form tX, applied formulas are restricted to be propositional variables, i.e., atomic formulas.

We denote propositional variables by capital letters $X, Y, Z, \ldots$, and other term-variables by small letters $x, y, z, \ldots$. By $FV(t)$ (and $FV(A)$), we denote the set of propositional variables as well as term-variables those freely appear in a term t (resp. in a formula A).

We consider the usual $\beta\eta$-reduction rules for proof-terms.

Definition 2.3 $\beta\eta$-reduction rules are as follows.

β-reduction: $(\lambda x^A.t)s \to t[x := s]$ and $(\Lambda X.t)Y \to t[X := Y]$
η-reduction: $\lambda x^A.(tx) \to t$ when $x \not\in FV(t)$ and $\Lambda X.(tX) \to t$ when $X \not\in FV(t)$

We call each term of the left-hand side of reduction rules $\to$ as a **redex**. A term t is in **normal form**, if t contains no redex.

Thus, by normal form, we mean $\beta\eta$-normal form in what follows.

The following investigation of our phase semantics and proof-theoretic semantics can be exploited based on the above notion of reduction relation on terms. However, the slightly weaker notion of equality relation on terms makes our definitions and discussion much simpler. Thus, to make the outline of our discussion clear, we mainly consider the following $\beta\eta$-equality relation on terms.

Definition 2.4 $\beta\eta$-equality relation $\simeq$ is the reflexive, symmetric, and transitive relation on terms satisfying the following conditions:

- $(\lambda x^A.t)s \simeq t[x := s]$, and $(\Lambda X.t)Y \simeq t[X := Y]$.
- $\lambda x^A.(tx) \simeq t$ when $x \not\in FV(t)$, and $\Lambda X.(tX) \simeq t$ when $X \not\in FV(t)$.
- If $s \simeq t$, then $us \simeq ut$; $su \simeq tu$; $sX \simeq tX$; $\lambda x^A.s \simeq \lambda x^A.t$; and $\Lambda X.s \simeq \Lambda X.t$.

Inference rules of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ are the usual second-order natural deduction rules for → and ∀, except for ∀-elimination rule, whose instantiation is restricted to a propositional variable.

Definition 2.5 (**Inference rules of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$**) A statement is of the form $t : A$ with a term t and a formula A. A **context** is a finite set of statements such that $x_1 : A_1, \ldots, x_n : A_n$ where all $x_1, \ldots, x_n$ are distinct variables. We write $\Gamma, \Delta, \ldots$ for any context. **A term t is a proof of A from assumptions** Γ if a sequent $\Gamma \vdash t : A$ is derivable by the following inference rules of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$.

$$\frac{}{\Gamma, x : A \vdash x : A}\,ax$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \to B}\to i \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash s : A}{\Gamma \vdash ts : B}\to e$$

$$\frac{\Delta \vdash t : A}{\Delta \vdash \Lambda X.t : \forall X.A}\,\forall i \qquad \frac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash tY : A[X := Y]}\,\forall e$$

where $X \not\in FV(\Delta)$.

$\to i$ and $\forall i$ are introduction rules, and $\to e$ and $\forall e$ are elimination rules.

2.2 Phase semantics for $\mathsf{IL}^2_{\to\forall}\mathsf{at}$

Phase semantics was introduced by Girard (Girard 1987) as the usual set-theoretical semantics for linear logic. Based on the domain consisting of a commutative monoid M, each formula is interpreted as a closed subset $\alpha \in \mathcal{P}(M)$ satisfying certain topological closure conditions. In particular, our connectives→and ∀ are interpreted as follows.

- $(A \to B)^* = \{ m \in M \mid m \cdot n \in B^* \text{ for any } n \in A^* \}$
- $(\forall X.A)^* = \{ m \in M \mid m \in (A[\mathrm{X} := \mathrm{Y}])^* \text{ for any variable } Y \}$

See, for example, (Okada 2002) for phase semantics of classical and intuitionistic linear logic. Phase semantics for (non-linear) intuitionistic logic is obtained by imposing the idempotency condition on the underlying monoid, which corresponds to the contraction rule of Gentzen's sequent calculus, and by further imposing the monotonicity condition on closed sets, which corresponds to the weakening rule. In (Okada & Takemura

2007), phase semantics for intuitionistic logic is extended by augmenting the usual λ-terms, where elements of the monoid correspond to contexts, and the domain consists of the pairs of a context and a term. Then the completeness with respect to such extended phase semantics implies the normal form theorem of intuitionistic logic.

The following phase semantics is essentially the same as (Okada & Takemura 2007), although the atomic second-order $\forall$ is introduced and the Church-style system is adopted to capture the second-order structure of proofs.

Definition 2.6 A **phase space** $D_{\mathcal{M}}$ consists of the following items:

- A commutative monoid $\mathcal{M} = (M, \cdot, \varepsilon)$ that is idempotent, i.e., $m \cdot m = m$ for any $m \in M$. Thus, $\mathcal{M}$ is in fact a set.
- The domain of the space is $B_{\mathcal{M}} = \{(m \rhd t) \mid m \in M$, and t is a term$\}$.
- The set of closed sets $D_{\mathcal{M}} \subseteq \mathcal{P}(B_{\mathcal{M}})$ whose element α, called a **closed set**, satisfies the following closure conditions:

Monotonicity: If $(m \rhd t) \in \alpha$, then $(m \cdot n \rhd t) \in \alpha$ for any $n \in M$.
Equality: If $(m \rhd t) \in \alpha$ and $s \simeq t$, then $(m \rhd s) \in \alpha$.

Definition 2.7 A **phase model** $(D_{\mathcal{M}}, *)$ consists of a phase space $D_{\mathcal{M}}$ and an interpretation $*$ such that $X^* \in D_{\mathcal{M}}$.

The interpretation $*$ is extended to complex formulas as follows:

- $(A \to B)^* = \{(m \rhd t) \mid (m \cdot n \rhd ts) \in B^*$ for any $(n \rhd s) \in A^*\}$
- $(\forall X.A)^* = \{(m \rhd t) \mid (m \rhd tY) \in (A[X := Y])^*$ for any variable $Y\}$

Note that our connectives are interpreted based on corresponding elimination rules.
We usually denote $m \cdot n$ simply by mn.
It is shown that $(A \to B)^*$ and $(\forall X.A)^*$ are closed if so are A^* and B^*.

Lemma 2.8 (Closed) *If* $A^*, B^* \in D_{\mathcal{M}}$, *then* $(A \to B)^*, (\forall X.A)^* \in D_{\mathcal{M}}$.

Proof. To show Equality of $(A \to B)^*$, let $(m \rhd t) \in (A \to B)^*$. By definition, we have $(mn \rhd tu) \in B^*$ for any $(n \rhd u) \in A^*$. If $s \simeq t$, then we have $su \simeq tu$, and $(mn \rhd su) \in B^*$ since B^* is closed. Thus, we have $(m \rhd s) \in (A \to B)^*$.

To show Monotonicity, let $(m \rhd t) \in (A \to B)^*$. By definition, for any $(l \rhd s) \in A^*$, we have $(ml \rhd ts) \in B^*$ which implies $(mnl \rhd ts) \in B^*$ for any $n \in M$ since

B^* is closed. Thus, we have $(mn \triangleright t) \in (A \to B)^*$.

To show Equality of $(\forall X.A)^*$, let $(m \triangleright t) \in (\forall X.A)^*$. By definition, we have $(m \triangleright tY) \in (A[X := Y])^*$ for any variable Y. If $s \simeq t$, then we have $sY \simeq tY$, and $(m \triangleright sY) \in (A[X := Y])^*$ since $(A[X := Y])^*$ is closed. Thus, we have $(m \triangleright s) \in (\forall X.A)^*$.

To show Monotonicity, let $(m \triangleright t) \in (\forall X.A)^*$. By definition, for any variable Y, we have $(m \triangleright tY) \in (A[X := Y])^*$, which implies $(mn \triangleright tY) \in (A[X := Y])^*$ for any $n \in M$ since $(A[X := Y])^*$ is closed. Thus, we have $(mn \triangleright t) \in (\forall X.A)^*$. ■

Thus, every interpretation A^* is a closed set in any phase model.

Our notion of validity is defined for proof-terms. In the following definition, we assume all open assumptions of t are among $x_1 : A_1, \ldots, x_k : A_k$.

Definition 2.9 (Validity) A term t of B with assumptions $x_1 : A_1, \ldots, x_k : A_k$, i.e., $x_1 : A_1, \ldots, x_k : A_k \vdash t : B$, is **valid** in a phase model $(D_{\mathcal{M}}, *)$ if and only if, for any $(m_i \triangleright t_i) \in A_i^*$,

$$(\prod_{1 \leq i \leq k} m_i \triangleright t[x_1 := t_1, \ldots, x_k := t_k]) \in B^* \text{ in } (D_{\mathcal{M}}, *) .$$

$x_1 : A_1, \ldots, x_k : A_k \vdash t : B$ is valid if it is valid in any phase model.

In what follows, we abbreviate the simultaneous substitution $t[x_1 := t_1, \ldots, x_k := t_k]$ as $t[x_i \vec{:=} t_i]$.

2.3 Soundness and completeness of $\mathrm{IL}^2_{\to\forall}$at

We prove the soundness theorem of $\mathrm{IL}^2_{\to\forall}$at. Since our connectives are interpreted based on elimination rules in our phase semantics, the soundness of each elimination rule is immediate by the definition of the interpretation. On the other hand, the soundness of each introduction rule is derived from the equality closedness of the closed set.

Theorem 2.10 (Soundness) *If* $\Gamma \vdash t : A$ *is derivable in* $\mathrm{IL}^2_{\to\forall}$at, *then* $\Gamma \vdash t : A$ *is valid in any phase model.*

Proof. We assume Γ is $x_1 : A_1, \ldots, x_k : A_k$. We show the theorem by induction on the length of proofs as usual.

- When $\overline{\Gamma, x : A \vdash x : A}\,ax$, we show $(\prod m_i \cdot m \rhd x[x := t]) \in A^*$ for any $(m_i \rhd t_i) \in A_i^*$ and any $(m \rhd t) \in A^*$. This is obtained by Monotonicity of A^*.

- When $\dfrac{\Gamma \vdash t : A \to B \quad \Gamma \vdash s : A}{\Gamma \vdash ts : B} \to e$, by the induction hypothesis, for any $(m_i \rhd t_i) \in A_i^*$, we have $(\prod m_i \rhd t[x_i := t_i]) \in (A \to B)^*$ and $(\prod m_i \rhd s[x_i \vec{:=} t_i]) \in A^*$. Then by the definition of the interpretation of the implication, we have $(\prod m_i \rhd ts[x_i \vec{:=} t_i]) \in B^*$.

- When $\dfrac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A.t : A \to B} \to i$, by the induction hypothesis, for any $(m_i \rhd t_i) \in A_i^*$ and any $(m \rhd s) \in A^*$, we have $(\prod m_i \cdot m \rhd t[x_i \vec{:=} t_i, x := s]) \in B^*$. Since $t[x_i \vec{:=} t_i, x := s] \simeq (\lambda x^A.t\ [x_i \vec{:=} t_i])\ s$ and B^* is equality closed, we have $(\prod m_i \cdot m \rhd (\lambda x^A.t[x_i \vec{:=} t_i])s) \in B^*$. Thus, $(\prod m_i \rhd \lambda x^A.t[x_i \vec{:=} t_i]) \in (A \to B)^*$.

- When $\dfrac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash tY : A[X := Y]} \forall e$, by the induction hypothesis, for any $(m_i \rhd t_i) \in A_i^*$, we have $(\prod m_i \rhd t[x_i \vec{:=} t_i]) \in (\forall X.A)^*$. Then by the definition of $\forall$, we have $(\prod m_i \rhd tY[x_i \vec{:=} t_i]) \in (A[X := Y])^*$.

- When $\dfrac{\Gamma \vdash t : A}{\Gamma \vdash \Lambda X.t : \forall X.A} \forall i$, where $X \not\in FV(\Gamma)$, by the induction hypothesis, for any $(m_i \rhd t_i) \in A_i^*$, we have $(\prod m_i \rhd t[x_i \vec{:=} t_i]) \in A^*$. Thus, we also have $(\prod m_i \rhd t[x_i \vec{:=} t_i, X := Y]) \in (A[X := Y])^*$, which is proved by induction on A. Since $t[x_i \vec{:=} t_i, X := Y] \simeq (\Lambda X.t)Y[x_i \vec{:=} t_i]$ and $(A[X := Y])^*$ is equality closed, we have $(\prod m_i \rhd (\Lambda X.t)Y[x_i \vec{:=} t_i]) \in (A[\mathrm{X} := \mathrm{Y}])^*$. Therefore, $(\prod m_i \rhd \Lambda X.t[x_i \vec{:=} t_i]) \in (\forall X.A)^*$. ■

To show the completeness theorem, we construct *canonical model*, i.e. , a syntactical model, where the validity in this model implies the derivability, as well as existence of a normal form.

Definition 2.11 $(D_{\mathcal{C}}, *)$ is constructed as follows:

- A commutative monoid $\mathcal{C}$ that consists of all contexts with the empty sequent $\emptyset$

as its unit.

- $B_C = \{(\Gamma \rhd t) \mid \Gamma \in \mathcal{C}$, and t is a term $\}$.
- $[\![A]\!] = \{(\Gamma \rhd t) \mid t \simeq s$ such that $\Gamma \vdash s : A$ is derivable in $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ and s is in normal form $\}$.
- $X^* = [\![X]\!]$.

Lemma 2.12 (Canonical model) *$(D_C, *)$ is a phase model.*

Proof. It is sufficient to show that $[\![A]\!]$ is closed.

To show Equality, let $(\Gamma \rhd t) \in [\![A]\!]$. By definition of $[\![A]\!]$, there exists a normal form u such that $t \simeq u$ and $\Gamma \vdash u : A$ is derivable. Thus, if $s \simeq t$, we have $s \simeq t \simeq u$ and $\Gamma \vdash u : A$ isderivable. Therefore, we have $(\Gamma \rhd s) \in [\![A]\!]$. Monotonicity is immediate from the derived rule of weakening in $\mathsf{IL}^2_{\to\forall}\mathsf{at}$.

Thus, $X^* (= [\![X]\!])$ belongs to the set of closed sets D_C, and $(D_C, *)$ is a phase model. ■

In the canonical model $(D_C, *)$, the following main lemma holds.

Lemma 2.13 (Main lemma) *$(x : A \rhd x) \in A^* \subseteq [\![A]\!]$ holds in $(D_C, *)$.*

Proof. We divide this main lemma into the following two claims:

1. If $(\Gamma \rhd xT_1 \cdots T_n) \in [\![A]\!]$, then $(\Gamma \rhd xT_1 \cdots T_n) \in A^*$,
   where $xT_1 \cdots T_n$ is the abbreviation of applications $(\cdots((xT_1)T_2)\cdots)T_n$, and each T_i denotes a term or a propositional variable.
2. $A^* \subseteq [\![A]\!]$

The first half of the main lemma $(x : A \rhd x) \in A^*$ is obtained from the above claim (1), since $(x : A \rhd x) \in [\![A]\!]$. We show the above two claims by induction on the complexity of A.

- When $A \equiv X$, since $X^* = [\![X]\!]$ by definition, our claims (1) and (2) hold immediately.
- When $A \equiv B \to C$,

(1) assume $(\Gamma \rhd xT_1 \cdots T_n) \in [\![B \to C]\!]$. To show $(\Gamma \rhd xT_1 \cdots T_n) \in (B \to C)^*$, let $(\Delta \rhd t) \in B^*$. Then, by the induction hypothesis, we have $B^* \subseteq [\![B]\!]$, and hence, we have $t \simeq v$ for some normal v such that $\Delta \vdash v : B$ is derivable.

On the other hand, by the assumption $(\Gamma \rhd xT_1 \cdots T_n) \in [\![B \to C]\!]$, we have $xT_1 \cdots T_n \simeq u$ for some normal u such that $\Gamma \vdash u : B \to C$ is derivable. Note that we find the normal u is of the form $xT'_1 \cdots T'_n$, where T'_i is a normal term or a propositional variable. Thus, $(xT_1 \cdots T_n)t \simeq uv$ and uv is in normal form. Furthermore, we have

$$\frac{\Gamma \vdash u : B \to C \quad \Delta \vdash v : B}{\Gamma, \Delta \vdash uv : C} \to e$$

Thus, we have $(\Gamma \cdot \Delta \rhd (xT_1 \cdots T_n)t) \in [\![C]\!]$, and hence, by the induction hypothesis, we have $(\Gamma \cdot \Delta \rhd (xT_1 \cdots T_n)t) \in C^*$. Therefore, $(\Gamma \rhd xT_1 \cdots T_n) \in (B \to C)^*$.

(2) To show $(B \to C)^* \subseteq [\![B \to C]\!]$, assume $(\Gamma \rhd t) \in (B \to C)^*$. Then, for any $(\Delta \rhd s) \in B^*$, we have $(\Gamma \cdot \Delta \rhd ts) \in C^*$. In particular, we have $(\Gamma \cdot x : B \rhd tx) \in C^*$ since $(x : B \rhd x) \in B^*$ by the induction hypothesis. Here, we assume $x \not\in FV(t)$ without loss of generality. Again by the induction hypothesis $C^* \subseteq [\![C]\!]$, we have $tx \simeq u$ for some normal u such that $\Gamma, x : B \vdash u : C$ is derivable. $tx \simeq u$ implies $t \simeq \lambda x^B. (tx) \simeq \lambda x^B.u$. If (i) u is of the form vx with $x \not\in FV(v)$, $\lambda x^B.u$ is not normal, but (ii) otherwise, $\lambda x^B.u$ is normal and, in this case (ii), we have

$$\frac{\Gamma, x : B \vdash u : C}{\Gamma \vdash \lambda x^B.u : B \to C} \to i$$

In the first case (i) , we have $t \simeq \lambda x^B.(vx) \simeq v$, where v is normal since vx is so. Furthermore, since $\Gamma, x : B \vdash vx : C$ is derivable and $x \not\in FV(v)$, we find $\Gamma \vdash v : B \to C$ is derivable.

Therefore, $(\Gamma \rhd t) \in [\![B \to C]\!]$.

• When $A \equiv \forall X.B$,

(1) assume $(\Gamma \rhd xT_1 \cdots T_n) \in [\![\forall X.B]\!]$. Then, we have $xT_1 \cdots T_n \simeq u$ for some normal u such that $\Gamma \vdash u : \forall X.B$ is derivable. Observe that the normal form u is of the form $xT'_1 \cdots T_n'$, where T'_i is a normal term or a propositional variable. Hence, we have $(xT_1 \cdots T_n)\ Y \simeq uY$ and uY is normal for any variable Y. Furthermore, we have

$$\frac{\Gamma \vdash u : \forall X.B}{\Gamma \vdash uY : B[X := Y]} \forall e$$

Thus, we have $(\Gamma \rhd (xT_1 \cdots T_n)Y) \in [\![B[X := Y]]\!]$, and hence $(\Gamma \rhd (xT_1 \cdots T_n)Y) \in (B[X := Y])^*$ by the induction hypothesis. Therefore, we have $(\Gamma \rhd xT_1 \cdots T_n) \in (\forall X.B)^*$.

(2) To show $(\forall X.B)^* \subseteq [\![\forall X.B]\!]$, assume $(\Gamma \rhd t) \in (\forall X.B)^*$. Without loss of generality, we assume $X \not\in FV(t) \cup FV(\Gamma)$. Then, by definition, we have $(\Gamma \rhd tY) \in (B[X := Y])^*$ for any Y. In particular, we have $(\Gamma \rhd tX) \in B^*$, and then, by the induction hypothesis $B^* \subseteq [\![B]\!]$,we have $(\Gamma \rhd tX) \in [\![B]\!]$. Thus, we have $tX \simeq u$ for some normal u such that $\Gamma \vdash u : B$ is derivable. Then, from $tX \simeq u$, we have $t \simeq \Lambda X.(tX) \simeq \Lambda X.u$. If (i) u is of the form vX with $X \not\in FV(v)$, then $\Lambda X.u$ is not nor-

mal, but (ii) otherwise, $\Lambda X.u$ is normal and, in this case (ii),we have

$$\frac{\Gamma \vdash u : B}{\Gamma \vdash \Lambda X.u : \forall X.B}\ \forall i$$

In the first case (i), we have $t \simeq \Lambda X.(vX) \simeq v$, where v is normal since vX is so. Furthermore, since $\Gamma \vdash vX : B$ is derivable and $X \not\in FV(v)$, we find $\Gamma \vdash v : \forall X.B$ is derivable.

Therefore, $(\Gamma \rhd t) \in [\![\forall X.B]\!]$. ■

Theorem 2.14 (Completeness) *If $\Gamma \vdash t : A$ is valid in any phase model, then there exists a normal form s such that $t \simeq s$ and $\Gamma \vdash s : A$ is derivable in* $\mathsf{IL}^2_{\to\forall}\mathsf{at}$.

Proof. Let $\Gamma \vdash t : A$ is valid in any phase model. Then, in particular, $\Gamma \vdash t : A$ is valid in the canonical model $(D_C, *)$, that is, for any $(\Gamma_i \rhd t_i) \in A_i^*$, we have $(\prod \Gamma_i \rhd t[x_1 := t_1, \ldots, x_k := t_k]) \in A^*$. Since $(x_i : A_i \rhd x_i) \in A_i^*$ in $(D_C, *)$, we have $(\Gamma \rhd t) \in A^*$, which implies $(\Gamma \rhd t) \in [\![A]\!]$ since $A^* \subseteq [\![A]\!]$ in $(D_C, *)$. Therefore, there exists a normal form s such that $t \simeq s$ and $\Gamma \vdash s : A$ is derivable. ■

Although the above completeness is slightly weaker than our soundness (Theorem 2.10), if we add the following rule to $\mathsf{IL}^2_{\to\forall}\mathsf{at}$, the validity completely corresponds to the derivability.

$$\frac{\Gamma \vdash t : A \quad t \simeq s}{\Gamma \vdash s : A}$$

By combining our soundness and completeness, we obtain the following normal form theorem.

Corollary 2.15 (Normal form) *If $\Gamma \vdash t : A$ is derivable in* $\mathsf{IL}^2_{\to\forall}\mathsf{at}$, *then there exists a term s in normal form such that $s \simeq t$.*

3 Proof-theoretic semantics and phase semantics

We review proof-theoretic semantics of Prawitz and Schroeder-Heister in Section 3.1. Then, in Section 3.2, we give a characterization of proof-theoretic semantics in our phase semantic framework.

3.1 Proof-theoretic validity of Prawitz and Schroeder-Heister

By developing Gentzen's idea, Prawitz (Prawitz 1971) formally defined the proof-theoretic validity for deductions in the system of natural deduction for intuitionistic logic. A deduction is valid, if it is in canonical form that ends with an introduction rule, and if its immediate subdeductions are valid. A deduction ends with an elimination rule is valid if it reduces to the canonical form. See, for example, (Schroeder-Heister 2006; Piecha & Schroeder-Heister 2016; Schroeder-Heister 2012) for recent studies on proof-theoretic semantics.

3.1.1 Validity based on introduction rules: I-validity

Let us review, in more detail, the notion of proof-theoretic validity based on introduction rule, called I-validity here. The following definition is essentially that of (Schroeder-Heister 2006), although we generalize the reduction relation on terms to the equality relation. Furthermore, we do not consider any extension of basic atomic system. This is because such an extension is not considered in recent papers of Prawitz (cf. (Prawitz 1973; Prawitz 2013)), and the version without base extensions is more appropriate for our discussion.

Let S be an atomic system, called "atomic base", which is given by production rules for atomic formulas, and which fixes the validity of atomic formulas. By slightly abusing the notation, by S, we also denote a set of proof-terms consisting of production rules of the atomic system S. Then, based on the given atomic base S, the validity of proof-terms is defined. Since the notion of I-validity is defined with respect to a given atomic base S, it is more properly said that "a term t is I-valid relative to S". However, in what follows, we simply say "t is I-valid" when the given atomic base is clear from the context.

The notion of proof-theoretic validity is firstly defined for closed terms, which do not contain any free term-variable. Then, the validity of open terms containing free term-variables is defined by substituting valid closed terms for free term-variables appropriately. In what follows, we say a term t is "closed" if it does not contain any free term-variable.

1. A closed term t of an atom X is I-valid *iff* $t \simeq s$ such that $s \in S$.
2. A closed term t of $B \to C$ is I-valid *iff* $t \simeq \lambda x^B.u$ such that $u[x := s]$ of C is I-valid for every closed I-valid s of B.
3. A closed term t of $\forall X.A$ is I-valid *iff* $t \simeq \Lambda X.u$ such that $u[X := Y]$ of $A[X := Y]$ is I-valid for every atom Y.
4. An open term t of B from $A_1, \ldots, A_n$, where all open assumption of t are among

$A_1, \ldots, A_n$, is I-valid *iff* $t[x_i \vec{:=} t_i]$ of B is I-valid for every list of closed I-valid t_i of A_i $(1 \leq i \leq n)$.

The above definition (3) for the atomic second-order quantifier $\forall$ is based on that for the first-order $\forall$ given in (Prawitz 1971). The terms $s \in S$ of (1), $\lambda x^B.u$ of (2), and $\Lambda X.u$ of (3) are called "canonical forms" in Prawitz's original definition.

3.1.2 Validity based on elimination rules: E-validity

The duality between introduction and elimination rules of natural deduction enables us to define the notion of validity based on elimination rules. We call this "E-validity", and it is also discussed by Prawitz and Schroeder-Heister, for example, in (Prawitz 1971; Prawitz 2007; Schroeder-Heister 2006). Only the following cases (2) and (3) for complex formulas are different from those of I-validity, and they are defined based on the corresponding elimination rules.

1. A closed term t of an atom X is E-valid *iff* $t \simeq s$ such that $s \in S$.
2. A closed term t of $B \to C$ is E-valid *iff* ts of C is E-valid for every closed E-valid s of B.
3. A closed term t of $\forall X.A$ is E-valid *iff* tY of $A[X := Y]$ is E-valid for every atom Y.
4. An open term t of B from $A_1, \ldots, A_n$ is E-valid *iff* $t[x_i \vec{:=} t_i]$ of B is E-valid for every list of closed E-valid t_i of A_i $(1 \leq i \leq n)$.

See (Prawitz 2007) for philosophical discussion on differences between I-validity and E-validity.

3.2 Proof-theoretic semantics as phase semantics

We investigate proof-theoretic semantics of Prawitz and Schroeder-Heister in our phase semantic framework. We first give a set-theoretical description of proof-theoretic semantics. Let us begin with the E-validity, since its definition is closer to our interpretation of connectives in phase semantics.

3.2.1 E-validity in phase semantics

Let us replace "a closed term t of A is E-valid" given in Section 3.1.2 by $t \in A^*$. Then, the definition of E-validity is described as follows.

1. $t \in X^*$ *iff* $t \simeq s$ such that $s \in S$.
2. $t \in (B \to C)^*$ *iff* $ts \in C^*$ for every $s \in B^*$.
3. $t \in (\forall X.A)^*$ *iff* $tY \in (A[X := Y])^*$ for every atom Y.
4. $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ is E-valid *iff* $t[\vec{x_i := t_i}] \in B^*$ for every $t_i \in A_i^*$.

Note that the validity of open terms (4) is essentially the same as our Definition 2.9.

The above set-theoretical description suggests a phase model consisting only of closed terms. In particular, the interpretation of connectives $\to$ and $\forall$ is the same as that in phase semantics.

Firstly, in terms of syntax, we extend $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ by introducing an atomic base S: We add $\vdash t : X$ as an axiom of $\mathsf{IL}^2_{\to\forall}\mathsf{at}$ *with* S for every t of X that belongs to S.

Next, we construct a special model by using only closed terms as follows.

Definition 3.1 $(D_{\mathcal{E}}, *_S)$ is constructed as follows:

- A commutative monoid $\mathcal{E} = \{\emptyset\}$, which consists only of the empty sequent $\emptyset$.
- The domain is $B_{\mathcal{E}} = \{(\emptyset \rhd t) \mid t$ is a closed term $\}$.
- $X^* = [\![X]\!]_S = \{(\emptyset \rhd t) \mid t \simeq s$ such that $\vdash s : X$ belongs to $S\}$.

Although we usually denote $(\emptyset \rhd t)$ simply by t, note that t is a closed term in this context.

It is shown that every $[\![X]\!]_S$ is a closed set, and we find that $(D_{\mathcal{E}}, *_S)$ is a phase model.

Lemma 3.2 (Phase model for E-validity) $(D_{\mathcal{E}}, *_S)$ *is a phase model.*

Proof. It is sufficient to show that $[\![X]\!]_S$ is closed. We show that, for any $s \in B_{\mathcal{E}}$, if $t \in [\![X]\!]_S$ and $s \simeq t$, then $s \in [\![X]\!]_S$. Let $t \in [\![X]\!]_S$ and $s \simeq t$. Then, by $t \in [\![X]\!]_S$, we have $t \simeq u$ for some $u \in S$ such that $\vdash u : X$ is derivable. Since $s \simeq t$ by assumption, we have $s \simeq t \simeq u$, and hence, $s \in [\![X]\!]_S$. Monotonicity of $[\![X]\!]_S$ is trivial since we do not consider any context here.

Thus, our $(D_{\mathcal{E}}, *_S)$ is a phase model. ■

Since $(D_{\mathcal{E}}, *_S)$ is just the phase semantic description of the E-validity of proof-theoretic semantics, the following is clear.

Proposition 3.3 $\Gamma \vdash t : A$ *is valid in* $(D_{\mathcal{E}}, *_S)$ *if and only if* $\Gamma \vdash t : A$ *is E-valid in proof-theoretic semantics.*

By the above proposition, we find the following relationship between our phase semantics and proof-theoretic semantics.

Proposition 3.4 *If* $\Gamma \vdash t : A$ *is valid in any phase model, then* $\Gamma \vdash t : A$ *is E-valid in proof-theoretic semantics.*

Proof. If $\Gamma \vdash t : A$ is valid in any phase model, then, in particular, it is valid in $(D_{\mathcal{E}}, *_S)$, which is equivalent to that $\Gamma \vdash t : A$ is E-valid in proof-theoretic semantics by Proposition 3.3. ■

The reverse of the above proposition implies the Prawitz's completeness conjecture, which is discussed in Section 4.

3.2.2 I-validity in phase semantics

Next, to investigate the notion of I-validity in our phase semantics, we construct a phase model for I-validity. The domain and the interpretation of atomic formulas are the same as those of the phase model for E-validity. We modify the interpretation of $\to$ and $\forall$. To distinguish from the previous interpretation $*$ for E-validity, we use $\bigstar$ for the interpretation of $\to$ and $\forall$ based on their introduction rules.

Definition 3.5 $(D_{\mathcal{E}}, \bigstar_S)$ is constructed as follows:

- The space $D_{\mathcal{E}}$ is the same as the phase space for E-validity of Definition 3.1.
- $X^\bigstar = [\![X]\!]_S$.
- $(B \to C)^\bigstar = \{t \mid t \simeq \lambda x^B.u$ such that $u[x := s] \in C^\bigstar$ for any $s \in B^\bigstar\}$.
- $(\forall X.A)^\bigstar = \{t \mid t \simeq \Lambda X.u$ such that $u[X := Y] \in (A[X := Y])^\bigstar$ for any $Y\}$.

Lemma 3.6 (Phase model for I-validity) $(D_{\mathcal{E}}, \bigstar_S)$ *is a phase model.*

Proof. To show this lemma, we prove $A^* = A^\bigstar$ by induction on A.
(1) When $A \equiv X$, we have $X^* = [\![X]\!]_S = X^\bigstar$ by definition.
(2) When $A \equiv B \to C$, we first show $(B \to C)^\bigstar \subseteq (B \to C)^*$. Let $t \in (B \to C)^\bigstar$. Then, by definition, $t \simeq \lambda x^B.u$ such that $u[x := s] \in C^\bigstar$ for any $s \in B^\bigstar$. Assume $s \in B^*$ ($= B^\bigstar$ by the induction hypothesis). Since $t \simeq \lambda x^B.u$, we have $ts \simeq (\lambda x^B.u)s \simeq u[x := s] \in C^\bigstar$, which implies $ts \in C^*$ by the induction hypothesis. Therefore, we have $t \in (B \to C)^*$.

Next, we show $(B \to C)^* \subseteq (B \to C)^\bigstar$. Let $t \in (B \to C)^*$. Since $(B \to C)^*$ is equality closed, we have $\lambda x^B.(tx) \simeq t \in (B \to C)^*$ for some $x \not\in FV(t)$. Thus, t is equivalent to a λ-abstraction term. Next, we show $tx[x := s] \in C^\bigstar$ for any $s \in B^\bigstar$. Let $s \in B^\bigstar$ ($= B^*$ by the induction hypothesis). Then, since $t \in (B \to C)^*$, we have $ts \in C^*$, and hence, $tx[x := s] \equiv ts \in C^* = C^\bigstar$ by the induction hypothesis. Therefore, $t \in (B \to C)^\bigstar$.

(3) When $A \equiv \forall X.B$, we first show $(\forall X.B)^\bigstar \subseteq (\forall X.B)^*$. Let $t \in (\forall X.B)^\bigstar$. Then, by definition, $t \simeq \Lambda X.u$ such that $u[X := Y] \in (B[X := Y])^\bigstar$ for any Y. Thus, for any Y, we have:

$$tY \simeq (\Lambda X.u)Y \simeq u[X := Y] \in (B[X := Y])^\bigstar = (B[X := Y])^*$$

Therefore, $tY \in (B[X := Y])^*$, and we have $t \in (\forall X.B)^*$.

Next, we show $(\forall X.B)^* \subseteq (\forall X.B)^\bigstar$. Let $t \in (\forall X.B)^*$. Since $(\forall X.B)^*$ is closed, we have $\Lambda Y.(tY) \simeq t \in (\forall X.B)^*$ with $Y \not\in FV(t)$. Thus, t is equivalent to a Λ-abstraction term. We then show $tY[Y := Z] \in (B[Y := Z])^*$ for any Z. Since $t \simeq \Lambda Y.(tY)$, we have:

$$tY[Y := Z] \equiv tZ \simeq (\Lambda Y.(tY))Z \in (B[Y := Z])^* = (B[Y := Z])^\bigstar$$

Therefore, we have $t \in (\forall X.B)^\bigstar$.

Thus, $(D_\varepsilon, \bigstar_S)$ is a phase model. ■

Thus, both proof-theoretic E-validity and I-validity are characterized in our phase semantics as phase models consisting only of closed terms.

4 Conclusion and discussion

If we consider a tightly typed system, where only typable terms, i.e., proofs are allowed to be terms legally, then Prawitz's conjecture is trivial. This is because in such a system, the conjecture just says that "valid proofs are proofs." Thus, Prawitz considered a broader class of reasoning called "arguments", which are essentially trees of formulas built from arbitrary inferences including invalid inferences. For example, as (Schroeder-Heister 2006) describes, the following inference may appear in an argument:

$$\frac{A \to (B \to C)}{B \to (A \to C)}$$

As well as valid inferences, invalid inferences such as the following inference may appear in an argument:

$$\frac{A \to B}{B \to A}$$

Since proof-theoretic validity is defined based on reduction rules on terms, according to the generalization to arguments, reduction rules are also generalized, which are called "justifications". Prawitz considers the completeness of such a system of arguments. Although our proof-terms are not so general as Prawitz's arguments, our system is not tightly typed and contains untypable terms, which may be considered as a certain kind of arguments.

If we extend the notion of Prawitz's validity so that it is applied to terms with contexts containing free variables, and if we regard $x : A \vdash x : A$ as valid for any formula A, then our phase semantics can be regarded as proof-theoretic semantics, and our completeness can be applied to such proof-theoretic semantics.

However, our completeness cannot be applied to the original Prawitz's proof-theoretic semantics straightforwardly. As discussed in the last section, for any closed term t of A, its validity in Prawitz's proof-theoretic semantics coincides with $t \in A^*$ in our phase model. Unfortunately, this correspondence cannot be extended to terms with free variables. Furthermore, our completeness proof also cannot be applied, since the fact $(x : A \rhd x) \in A^*$, i.e., $x : A \vdash x : A$ is valid, plays an essential role in our completeness proof.

Thus, if we remain the original proof-theoretic semantics of Prawitz and Schroeder-Heister, it is characterized, from the phase semantic viewpoint, as a phase model consisting only of closed terms. Then, Prawitz's completeness conjecture is characterized as the completeness with respect to the phase model consisting only of closed terms.

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要旨

「プラヴィッツの完全性予想について」

論理学における通常のタルスキ意味論での妥当性の概念は、文の真理条件に基づいて、文（論理式）に対して定義される。それに対してPrawitzとSchroeder-Heisterの証明論的意味での妥当性の概念は、証明の構成に基づいて、証明に対して定義される。Prawitzは、このような証明論的妥当性の概念を、妥当な推論のみで構成される証明だけではなく、非妥当な推論を含むより一般の「論証」に対して拡張し、証明論的意味論の完全性予想を提起した。本稿ではOkada & Takemura(2007)のPhase semanticsを用いて、証明論的意味論の特徴づけを与え、Phase semanticsの観点からPrawitzの完全性予想について考察する。