【Article】

Completeness of an Euler Diagrammatic System with Constant and Existential Points

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Abstract

We extend Euler diagrammatic system of Minshima, Okada, & Takemura (2012) by distinguishing constant and existential points. Constants points correspond to constant symbols, and existential points correspond to bound variables associated with the existential quantifier of the first-order language in symbolic logic. We prove the completeness theorem of our extended Euler diagrammatic system.

1 Introduction

Since the 1990s, reasoning with Euler and Venn diagrams has been studied from mathematical and logical viewpoint. Euler and Venn diagrams are rigorously defined as syntactic objects, like formulas in symbolic logic, and inference systems are formalized, which are equivalent to some symbolic logical systems. Then, fundamental logical properties such as soundness and completeness are investigated. E.g., (Shin, 1994; Howse et al., 2005; Mineshima et al., 2012). See (Stapleton, 2005) for a survey.

Mineshima, Okada, & Takemura (2012) introduced a basic Euler diagrammatic system, called GDS, in which a diagram consists of named circles and points. Then, the soundness and completeness theorems of the system are established. Furthermore, it is shown that the traditional Aristotelian categorical syllogisms are easily simulated in the system. In the simulation, an existential sentence such as "Some A are B" is translated into an Euler diagram through a sentence "c is A and c is B" for some constant c. Although this translation works well for the simulation of the syllogisms, if we consider the negation of a whole diagram beyond the syllogisms, the above translation may cause some problem: while "No A are B" is the negation of "Some A are B," it is no longer the negation of "c is A and c is B."

In this paper, we extend the Euler diagrammatic system of (Mineshima et al., 2012) by introducing existential points, which correspond to bound variables associated with the existential quantifier of the first-order language in symbolic logic. The introduction of existential points, distinguished from constant points, makes our Euler diagrammatic system more expressive, and enables us to translate syllogistic sentences more naturally.

In Section 2, we introduce the syntax, semantics, and inference rules of our Euler diagrammatic system. Then, in Section 3, by extending the proof given in (Mineshima et al., 2012), we prove the completeness theorem of our system.

2 System of Euler diagrams

Our Euler diagram, called EUL-diagram, is defined as a plane with named circles and points. Each diagram is specified by topological (inclusion and exclusion) relations maintained between circles and points. Thus diagrams are syntactically equivalent when the same relations hold on each. Based on the interpretation of circles (resp. points) as subsets (resp. elements) of a certain set-theoretical domain, diagrams are interpreted in terms of relations that hold on them. In Section 2.1, we introduce syntax and semantics of EUL-diagrams with existential points. In Section 2.2, we introduce our inference rules.

2.1 Syntax and semantics of EUL

Our Euler diagram in this paper is a slight extension of the most basic one of (Mineshima et al., 2012): As well as constant points, which correspond to constant symbols of the first order language FOL in symbolic logic, we here introduce existential points, which correspond to bound variables associated with the existential quantifier of FOL.

Definition 2.1. An **EUL-diagram** is a plane ($\mathbb{R}^2$) with a finite number, at least two, of **named simple closed curves** (simply called **named circles**, and denoted by $A, B, C, \dots$) and **constant points** (denoted by $a, b, c, \dots$), and **existential points** (denoted by $x, y, z, \dots$), where no two named circles, as well as points, are completely concurrent, and no two named circles, as well as points, have the same name.
Constant points and existential points are collectively called **(named) points**, and denoted by $p, q, p_1, p_2, \dots$. Named circles and points are collectively called **(diagrammatic) objects**, and denoted by $s, t, u, \dots$. We use a rectangle to represent the plane for a diagram. Diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \dots$.
When $\mathcal{D}$ is a diagram, we denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$.
A diagram consisting of only two objects is called a **minimal diagram**, and these are denoted by $\alpha, \beta, \gamma, \dots$.

Our diagrams are investigated in terms of the following topological relations between diagrammatic objects.

Definition 2.2. **EUL-relations** are the following reflexive asymmetric binary relation $\sqsubset$, and irreflexive symmetric binary relations $\vdash\!\dashv$ and $\bowtie$:

$A \sqsubset B$	"the interior of A is *inside of* the interior of B,"
$A \vdash\!\dashv B$	"the interior of A is *outside of* the interior of B,"
$A \bowtie B$	"there is at least one *crossing* point between A and B,"
$p \sqsubset A$	"p is *inside of* the interior of A,"
$p \vdash\!\dashv A$	"p is *outside of* the interior of A,"
$p \vdash\!\dashv q$	"p is *outside of* q (i.e. p is not located at the point of q)."

The set of EUL-relations that hold in a diagram $\mathcal{D}$ is uniquely determined, and we denote this set by $\mathsf{rel}(\mathcal{D})$. In the description of $\mathsf{rel}(\mathcal{D})$, we omit the trivial reflexive relation $s \sqsubset s$ for each object s.

We consider an equivalence class of plane diagrams in terms of the EUL-relations.

Definition 2.3. Any pair of EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ are **(syntactically) equivalent** if $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred by a single name. See (Mineshima et al., 2012), for a discussion about our slightly rough equation of diagrams.

Based on the above equivalence on diagrams, we refer to any minimal diagram, say α where $s \sqsubset t$ holds, by the non-trivial EUL-relation holding on it, as $s \sqsubset t$.

EUL-diagrams are interpreted in terms of EUL-relations that hold on them.

Definition 2.4. A **model** M is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function which assigns to each circle or constant point a *non-empty subset* of U such that $I(a)$ is a singleton for any constant point a, and $I(a) \neq I(b)$ for any constant points a, b of distinct names.

In order to avoid the complexity caused by existential points, we define our interpretation for *a set of diagrams* instead for a single diagram.

Definition 2.5. Let $\vec{\mathcal{D}}$ be a set of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ such that $\mathsf{rel}(\vec{\mathcal{D}}) = \mathsf{rel}(\mathcal{D}_1) \cup \cdots \cup \mathsf{rel}(\mathcal{D}_n) = \{R_1, \ldots, R_i, x_1 \Box A_1, \ldots, x_1 \Box A_k, \ldots, x_l \Box A_1, \ldots, x_l \Box A_k\}$, where $\Box$ is $\sqsubset$ or $\vdash\!\dashv$, and no existential points appear in each of $R_1, \ldots, R_i$.
$M = (U, I)$ is a model of $\vec{\mathcal{D}}$, written $M \models \vec{\mathcal{D}}$, if and only if:

- for every $R_j \in \mathsf{rel}(\vec{\mathcal{D}})$ $(1 \leq j \leq i)$,
  $I(s) \subseteq I(t)$ holds if R_j is $s \sqsubset t$; and
  $I(s) \cap I(t) = \emptyset$ holds if R_j is $s \vdash\!\dashv t$:
- for every x_j $(1 \leq j \leq l)$, there exists $m_j \in M$ such that $m_j \square I(A_1)$ and $\cdots$ and $m_j \square I(A_k)$ hold, where $\square$ is $\in$ if $x_j \sqsubset A \in \mathsf{rel}(\vec{\mathcal{D}})$, and it is $\notin$ if $x_j \vdash\!\dashv A \in \mathsf{rel}(\vec{\mathcal{D}})$.

From the symbolic logical viewpoint, the above interpretation corresponds to the usual set-theoretical interpretation of the following formula:

$$R_1 \wedge \cdots \wedge R_i \wedge \exists x_1((x_1 \square A_1) \wedge \cdots \wedge (x_1 \square A_k)) \wedge \cdots \wedge \exists x_l((x_l \square A_1) \wedge \cdots \wedge (x_l \square A_k))$$

Remark 2.6. By Definition 2.5, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $A \bowtie B$ might be understood as $I(A) \cap I(B) = \emptyset$ *or* $I(A) \cap I(B) \neq \emptyset$, which is true in any model.

The semantic consequence relation, $\models$ between EUL-diagrams is defined as usual in symbolic logic. (See (Mineshima et al., 2012) for a detailed description.)

2.2 Inference system GDS

Mineshima, Okada, & Takemura (2012) introduced an Euler diagrammatic inference system, called Generalized Diagrammatic Syllogistic inference system GDS. It consists of two kinds of inference rules: Deletion and Unification. Deletion allows us to delete a diagrammatic object from a given diagram. Unification (rules of U1–U10, PI) allows us to unify two diagrams into one diagram in which the semantic information is equivalent to the conjunction of the original two diagrams. Two kinds of constraint are imposed on unification. One is the *constraint for determinacy*, which blocks the disjunctive ambiguity with respect to locations of named points. The other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram.

Each unification rule is described by specifying (i) premise diagrams, one of which is required to be minimal; (ii) diagrammatic operations to introduce a new object into, or to rearrange a configuration of objects of, one of the premise diagrams. We also give schematic illustrations and concrete examples of applications of rules. We further specify the set of EUL-relations $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram.

In the following, we introduce our axiom, and of the eleven unification rules, we describe U5 and U7. The other unification rules are found in Appendix A and (Mineshima et al., 2012). We further introduce another rule of Ren (Renaming). In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{a}{\bullet}$, simply by its name a.

Definition 2.7 (Inference rules of GDS)**.**

Axiom:

$\boxed{\text{A1}}$ For any pair of circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

$\boxed{\text{A2}}$ Any EUL-diagram that consists only of points is an axiom.

$\boxed{\text{A3}}$ For any existential point x, and any circle A, any minimal diagram where $x \sqsubset A$ holds is an axiom.

Unification:

$\boxed{\text{U5}}$ Premises: A minimal diagram α on which $A \sqsubset B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$ (but $A \notin cr(\mathcal{D})$).

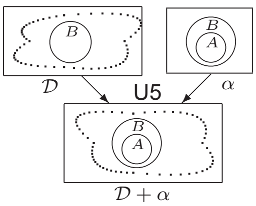
Constraint for determinacy: $p \vdash\!\dashv B$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $A \sqsubset B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $X \sqsubset B$ or $X \bowtie B$ holds on $\mathcal{D}$.

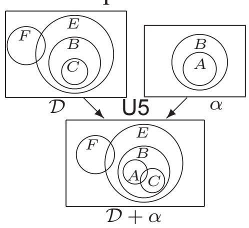
The set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash\!\dashv A \mid X \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\} \cup \{p \vdash\!\dashv A \mid p \in pt(\mathcal{D})\}$$

Schema of U5



Example of U5

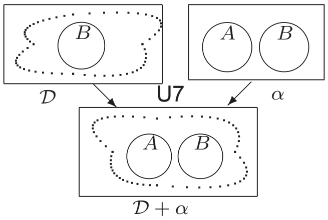


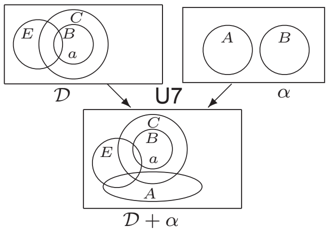
$\boxed{\text{U7}}$ Premises: A minimal diagram α on which $A \vdash\!\dashv B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$ (but $A \notin cr(\mathcal{D})$).

Constraint for determinacy: $p \sqsubset B$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $A \vdash\!\dashv B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $B \sqsubset X$ or $B \vdash\!\dashv X$ or $B \bowtie X$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid B \sqsubset X \text{ or } B \vdash\!\dashv X \text{ or } B \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{X \vdash\!\dashv A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \cup \{p \vdash\!\dashv A \mid p \in pt(\mathcal{D})\}$$





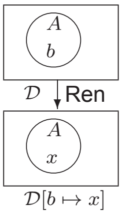
$\boxed{\text{Ren (Renaming)}}$ Premise: A diagram $\mathcal{D}$ containing a point p such that $p \not\equiv x$ for an existential point x.

Operation: Replace the point p of $\mathcal{D}$ with the existential point x.

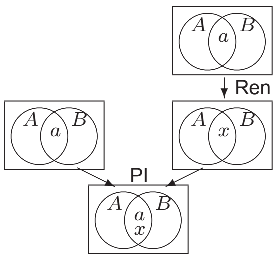
The set of relations $\mathsf{rel}(\mathcal{D}[p \mapsto x])$ of the resulting diagram is specified as follows:

$$(\mathsf{rel}(\mathcal{D}) \setminus \{p \Box s \mid \Box \in \{\sqsubset, \vdash\!\dashv\}, s \in ob(\mathcal{D})\}) \cup \{x \sqsubset s \mid p \sqsubset s \in \mathsf{rel}(\mathcal{D})\} \cup \{x \vdash\!\dashv s \mid p \vdash\!\dashv s \in \mathsf{rel}(\mathcal{D})\}$$

Example of Ren



Remark 2.8 (Copy)**.** Combining Ren and PI rules, we are able to duplicate any point in a diagram as illustrated in the following:



Note that, in general, an additional point is restricted to be existential, and it is located in the same region as the original point.

The notion of **diagrammatic proof** (or, **d-proof** for short) is defined inductively as tree structures consisting of Unification, Deletion, and Renaming steps. (Cf. Fig. 1 in Example 3.11.)

Definition 2.9. Let $\vec{\mathcal{D}}$ be a set of EUL-diagrams. An EUL-diagram $\mathcal{E}$ is **provable** from $\vec{\mathcal{D}}$, written as $\vec{\mathcal{D}} \vdash \mathcal{E}$, if there is a d-proof of $\mathcal{E}$ in GDS from $\mathcal{D}_1, \ldots, \mathcal{D}_m$ such that:

(1) $\mathcal{D}_i \in \vec{D}$; (2) U1, U2 rules (unification by a shared point) are not applied to any existential point provided by Ren rule; (3) the existential point of any axiom A3 appears only in diagrams directly follow the axiom.

Remark 2.10. In our system, we postulate the interpretation $I(A)$ of a circle A is non-empty. According to this definition, also in our inference rules, we adopt any minimal diagram such that $x \sqsubset A$ holds as an axiom (A3). Our definition corresponds to the existential import in the literature of syllogisms. Without this postulate, two diagrams $\mathcal{D}_1$ in that $A \sqsubset B$ holds, and $\mathcal{D}_2$ in that $A \vdash\!\dashv B$ holds, are consistent, when A denotes the empty set. (Cf. Lemma 3.3.) However, it is difficult to express $\mathcal{D}_1$ and $\mathcal{D}_2$ in a single diagram in our framework. Introduction of another device such as shading, which express the corresponding region is empty, may cope with this problem partly. However, the question remains whether we draw the shaded circle A inside B or outside B. Thus, in our basic Euler diagrammatic system, we assume the existential import.

Lemma 2.11. *The following hold in* GDS*:*

1. *If* $\vec{\mathcal{D}} \vdash u \sqsubset s$ *and* $\vec{\mathcal{D}} \vdash s \sqsubset t$*, then* $\vec{\mathcal{D}} \vdash u \sqsubset t$*;*
2. *If* $\vec{\mathcal{D}} \vdash u \sqsubset t$ *and* $\vec{\mathcal{D}} \vdash s \vdash\!\dashv t$*, then* $\vec{\mathcal{D}} \vdash u \vdash\!\dashv s$*;*
3. *If* $\vec{\mathcal{D}} \vdash u \sqsubset s$ *and* $\vec{\mathcal{D}} \vdash s \vdash\!\dashv t$ *and* $\vec{\mathcal{D}} \vdash v \sqsubset t$*, then* $\vec{\mathcal{D}} \vdash u \vdash\!\dashv v$*.*

3 Completeness of GDS

In this section, we prove soundness (Theorem 3.1) and completeness (Theorem 3.10) of GDS with respect to our formal semantics.

The soundness is shown by induction on the height of a given d-proof as usual.

Theorem 3.1 (Soundness)**.** *Let* $\vec{\mathcal{D}}, \mathcal{E}$ *be* EUL*-diagrams. If* $\mathcal{E}$ *is provable from the premises* $\vec{\mathcal{D}}$ *(*$\vec{\mathcal{D}} \vdash \mathcal{E}$*) in* GDS*, then* $\mathcal{E}$ *is an semantic consequence of* $\vec{\mathcal{D}}$ *(*$\vec{\mathcal{D}} \models \mathcal{E}$*).*

For the completeness, we impose the following condition for premise diagrams:

Definition 3.2. A set of diagrams $\vec{\mathcal{D}}$ is **consistent** if it has a model.

Without this condition, any diagram, say $\mathcal{E}$ where $A \vdash\!\dashv C$ holds, is a valid consequence of an inconsistent set of premise diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ where $a \sqsubset B$ and $a \vdash\!\dashv B$ hold, respectively, although there is no d-proof of $\mathcal{E}$ from $\mathcal{D}_1$ and $\mathcal{D}_2$ in GDS. This is because GDS does not have a rule that corresponds to the absurdity rule of usual natural deduction: we can infer any diagram from a pair of inconsistent diagrams.

It is obvious that the soundness theorem also holds under the assumption of the consistency of the premises. The following is an important consequence of the consistency:

Lemma 3.3 (Consistency)**.** *Let $\vec{\mathcal{D}}$ be a set of diagrams which is consistent. Then none of the following holds in* GDS *for any objects s and t:*

1. $\vec{\mathcal{D}} \vdash s \sqsubset t$ *and* $\vec{\mathcal{D}} \vdash s \bowtie t$.
2. *There is an object u such that* $\vec{\mathcal{D}} \vdash s \bowtie t$ *and* $\vec{\mathcal{D}} \vdash u \sqsubset s$ *and* $\vec{\mathcal{D}} \vdash u \sqsubset t$.

Proof. (1) By the soundness of GDS, in any model $M = (U, I)$ of $\vec{\mathcal{D}}$, we have $I(s) \subseteq I(t)$, and $I(s) \cap I(t) = \emptyset$. By the definition of our interpretation, we have $I(s) \neq \emptyset$, and hence, we have $I(s) \cap I(t) \neq \emptyset$, which contradicts to $I(s) \cap I(t) = \emptyset$.
(2) Assume u is a named point. Then, by the soundness of GDS, in any model $M = (U, I)$ of $\vec{\mathcal{D}}$, we have $I(s) \cap I(t) = \emptyset$ and $I(s) \cap I(t) \neq \emptyset$, but it is impossible. When u is a circle, by the definition of our interpretation, $I(u) \neq \emptyset$. Hence, $I(u) \subseteq I(s)$ and $I(u) \subseteq I(t)$ imply $I(s) \cap I(t) \neq \emptyset$. Thus, this case is also impossible as the previous case. ■

In order to show the completeness theorem of GDS, we construct two kinds of syntactic models, called *canonical models*, in a similar way as the construction of Lindenbaum algebras in the literature of algebraic semantics for various propositional logics.

We first define the simpler one.

Definition 3.4 (Canonical model $M_{\vec{\alpha}}$)**.** Let $\vec{\alpha}$ be a set of minimal diagrams which is consistent. A canonical model $M_{\vec{\alpha}} = (M_{\vec{\alpha}}, I_{\vec{\alpha}})$ for $\vec{\alpha}$ is defined as follows:

– The domain $M_{\vec{\alpha}}$ is the set of diagrammatic objects (named circles and points) which occur in any minimal diagram $\alpha \in \vec{\alpha}$.

– $I_{\vec{\alpha}}$ is an interpretation function such that, for any object t,

$$I_{\vec{\alpha}}(t) = \{s \mid \vec{\alpha} \vdash s \sqsubset t \text{ in } \mathsf{GDS}\} \cup \{t\}.$$

Observe that in the above definition of $I_{\vec{\alpha}}$, when t is a named point, say a, its interpretation $I_{\vec{\alpha}}(a)$ is the singleton $\{a\}$ since $\vec{\alpha} \not\vdash s \sqsubset a$ for any object s by soundness.

Lemma 3.5 (Canonical model $M_{\vec{\alpha}}$)**.** *Let $\vec{\alpha}$ be a set $\alpha_1, \dots, \alpha_n$ of minimal diagrams which is consistent. Then $M_{\vec{\alpha}}$ is a model of $\vec{\alpha}$.*

Proof. We divide into the following cases according to the relation $R \in \mathsf{rel}(\vec{\alpha})$. Since the case R is of the form $s \bowtie t$ is obvious, we assume $R \not\equiv s \bowtie t$ in what follows.
1. When R is of the form $s \sqsubset t$, in which s is not an existential point, we have $\vec{\alpha} \vdash s \sqsubset t$ in GDS since $s \sqsubset t \in \mathsf{rel}(\vec{\alpha})$. We show $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Let $u \in I_{\vec{\alpha}}(s)$.

(a) When $u \equiv s$, we immediately have $s \in I_{\vec{\alpha}}(t)$ by the fact $\vec{\alpha} \vdash s \sqsubset t$.

(b) Otherwise, by the definition of $I_{\vec{\alpha}}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By composing it with $\vec{\alpha} \vdash s \sqsubset t$ as seen in Lemma 2.11(1), we have $\vec{\alpha} \vdash u \sqsubset t$ in GDS, that is, $u \in I_{\vec{\alpha}}(t)$.

2. When R is of the form $s \vdash\!\dashv t$, in which s nor t is an existential point, we have $\vec{\alpha} \vdash s \vdash\!\dashv t$ in GDS. We show $M_{\vec{\alpha}} \models s \vdash\!\dashv t$, i.e., $I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t) = \emptyset$. When both s and t are constant points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha}}(s) \cap I_{\vec{\alpha}}(t)$.

(a) When $u \equiv s$, we have $s \in I_{\vec{\alpha}}(t)$, i.e., $\vec{\alpha} \vdash s \sqsubset t$. This, together with $\vec{\alpha} \vdash s \vdash\!\dashv t$, is a contradiction by Lemma 3.3(1). The same applies to the case $u \equiv t$.

(b) Otherwise, $s \not\equiv u \not\equiv t$, we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$ by the definition of $I_{\vec{\alpha}}(s)$ and $I_{\vec{\alpha}}(t)$. They contradict $\vec{\alpha} \vdash s \vdash\!\dashv t$ by Lemma 3.3(2).

3. When all relations of an existential point x of $\mathsf{rel}(\vec{\alpha})$ are $x \sqsubset A_1, \dots, x \sqsubset A_i, x \vdash\!\dashv B_1, \dots, x \vdash\!\dashv B_j$, we show that there exists $m \in M_{\vec{\alpha}}$ such that $m \in I_{\vec{\alpha}}(A_1) \cap \cdots \cap I_{\vec{\alpha}}(A_i) \cap \overline{I_{\vec{\alpha}}(B_1)} \cap \cdots \cap \overline{I_{\vec{\alpha}}(B_j)}$, where $\overline{X}$ denotes the complement of a set X.

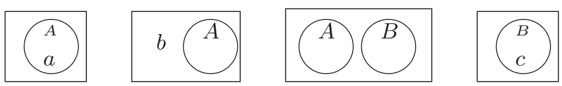
For every $1 \le k \le i$, we have $\vec{\alpha} \vdash x \sqsubset A_k$, since $x \sqsubset A_k \in \mathsf{rel}(\vec{\alpha})$. Hence, by the definition of $I_{\vec{\alpha}}$, we have $x \in I_{\vec{\alpha}}(A_k)$.

For every $1 \le l \le j$, to show $x \notin I_{\vec{\alpha}}(B_l)$, we assume to the contrary that $x \in I_{\vec{\alpha}}(B_l)$. Then, by the definition of $I_{\vec{\alpha}}$, we have $\vec{\alpha} \vdash x \sqsubset B_l$. On the other hand, since $x \vdash\!\dashv B_l \in \mathsf{rel}(\vec{\alpha})$, we have $\vec{\alpha} \vdash x \vdash\!\dashv B_l$. They are contradiction by Lemma 3.3(1).

Thus, we have $x \in I_{\vec{\alpha}}(A_1) \cap \cdots \cap I_{\vec{\alpha}}(A_i) \cap \overline{I_{\vec{\alpha}}(B_1)} \cap \cdots \cap \overline{I_{\vec{\alpha}}(B_j)}$. ■

As an illustration of the canonical model, let us consider the following example.

Example 3.6. Let $\vec{\alpha}$ be the following minimal diagrams $\alpha_1, \alpha_2, \alpha_3, \alpha_4$:



α_1 α_2 α_3 α_4

Observe that we have $\vec{\alpha} \not\vdash b \sqsubset B$ and $\vec{\alpha} \not\vdash b \vdash\!\dashv B$. In such a case, we say that the point b is indeterminate with respect to the circle B. Let us construct the canonical model for the $\vec{\alpha}$ by defining: $I_{\vec{\alpha}}(A) = \{A, a, \quad x, y, z, \dots\}$, where $x, y, z, \dots$ denotes all existential points, and $I_{\vec{\alpha}}(B) = \{B, c, \quad x, y, z, \dots\}$. Note that the indeterminate point b w.r.t. B is not contained in the interpretation of B. With this interpretation, for any named point $p \in I_{\vec{\alpha}}(B)$, we have $\vec{\alpha} \vdash p \sqsubset B$. In general, validity of $\sqsubset$-relation in the model $M_{\vec{\alpha}}$ implies provability of $\sqsubset$-relation.

In the above model, however, $p \notin I_{\vec{\alpha}}(B)$ does not necessarily imply $\vec{\alpha} \vdash p \vdash\!\dashv B$; because we do not have $\vec{\alpha} \vdash b \vdash\!\dashv B$, while $b \notin I_{\vec{\alpha}}(B)$. Thus, in the canonical model $M_{\vec{\alpha}}$ of Definition 3.4, validity of $\vdash\!\dashv$-relation does not imply provability of $\vdash\!\dashv$-relation, and hence the model is not enough to establish completeness. Let us try to modify the above model $M_{\vec{\alpha}}$ so that the indeterminate point b w.r.t. B is contained in the interpretation $I'_{\vec{\alpha}}(B)$ of B: $I'_{\vec{\alpha}}(A) = \{A, a, \quad x, y, z, \dots\}$ and $I'_{\vec{\alpha}}(B) = \{B, c, b, \quad x, y, z, \dots\}$. This definition also provides a model of $\vec{\alpha}$, and we have $\vec{\alpha} \vdash p \vdash\!\dashv B$ for any point $p \notin I'_{\vec{\alpha}}(B)$. However, in this model, $p \in I'_{\vec{\alpha}}(B)$ does not necessarily imply $\vec{\alpha} \vdash p \sqsubset B$; because we do not have $\vec{\alpha} \vdash b \sqsubset B$, while $b \in I'_{\vec{\alpha}}(B)$.

Although one of our two canonical models alone is insufficient to establish completeness, we can obtain our completeness result in the following manner: we construct a canonical model $M_{\vec{\alpha}}$ (Definition 3.4 above) for validity of $\sqsubset$-relation, which implies provability of $\sqsubset$-relation, and a model $M_{\vec{\alpha},B}$ (Definition 3.7 below) for validity of $\vdash\!\dashv$-relation, which implies provability of $\vdash\!\dashv$-relation.

We now construct another canonical model. In contrast to the previous model $M_{\vec{\alpha}}$ of Definition 3.4, we include, for a fixed circle B, indeterminate objects w.r.t. B into the interpretation of the circle B in the following model $M_{\vec{\alpha},B}$.

Definition 3.7 (Canonical model $M_{\vec{\alpha},B}$)**.** Let $\vec{\alpha}$ be a set of minimal diagrams which is consistent. Let B be a fixed named circle. A canonical model $M_{\vec{\alpha},B} = (M_{\vec{\alpha},B}, I_{\vec{\alpha},B})$ for $\vec{\alpha}$ is defined as follows:

– The domain $M_{\vec{\alpha},B}$ is the same set as $M_{\vec{\alpha}}$ of Definition 3.4.

– $I_{\vec{\alpha},B}$ is an interpretation function defined as follows: For any object t,

when $t \equiv B$ or $\vec{\alpha} \vdash B \sqsubset t$ holds,

$$I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t) \cup \{s \mid \vec{\alpha} \nvdash B \sqsubset s \text{ and } \vec{\alpha} \nvdash s \sqsubset B \text{ and } \vec{\alpha} \nvdash s \vdash\!\dashv B\};$$

otherwise, $I_{\vec{\alpha},B}(t) = I_{\vec{\alpha}}(t)$.

Lemma 3.8 (Canonical model $M_{\vec{\alpha},B}$)**.** *Let $\vec{\alpha}$ be a set $\alpha_1, \dots, \alpha_n$ of minimal diagrams which is consistent. Let B be a fixed named circle. Then $M_{\vec{\alpha},B}$ is a model of $\vec{\alpha}$.*

Proof. We divide into the following cases according to the form of $R \in \mathsf{rel}(\vec{\alpha})$. We write $\vec{\alpha} \nvdash s\Box t$ when none of $\vec{\alpha} \vdash s \sqsubset t, \vec{\alpha} \vdash t \sqsubset s$, and $\vec{\alpha} \vdash s \vdash\!\dashv t$ holds.

Since the case R is of the form $s \bowtie t$ is trivial, we assume R is not of the form.

1. When R is of the form $s \sqsubset t$, we have $\vec{\alpha} \vdash s \sqsubset t$ since $s \sqsubset t \in \mathsf{rel}(\vec{\alpha})$. We show $I_{\vec{\alpha},B}(s) \subseteq I_{\vec{\alpha},B}(t)$. Let $u \in I_{\vec{\alpha},B}(s)$. Since the case $u \equiv s$ is immediate, we assume $u \not\equiv s$.

Assume $s \equiv B$ or $\vec{\alpha} \vdash B \sqsubset s$ hold. Then, by the definition of $I_{\vec{\alpha},B}(s)$, we have (i) $\vec{\alpha} \vdash u \sqsubset s$ or (ii) $\vec{\alpha} \nvdash u\Box B$. (i) implies, together with $\vec{\alpha} \vdash s \sqsubset t$, that $\vec{\alpha} \vdash u \sqsubset t$, by Lemma 2.11(1), i.e., $u \in I_{\vec{\alpha},B}(t)$. For (ii), $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash s \sqsubset t$ imply $\vec{\alpha} \vdash B \sqsubset t$ by Lemma 2.11(1). Hence, in conjunction with $\vec{\alpha} \nvdash u\Box B$, we have $u \in I_{\vec{\alpha},B}(t)$ by the definition of $I_{\vec{\alpha},B}$. The other case ($s \not\equiv B$ and $\vec{\alpha} \nvdash B \sqsubset s$) is similar.

2. When R is of the form $s \mapsto t$, we have $\vec{\alpha} \vdash s \mapsto t$ since $s \mapsto t \in \mathsf{rel}(\vec{\alpha})$. We assume $s \not\equiv B \not\equiv t$ and $s \not\equiv u \not\equiv t$ since the other cases are similar. We show that $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t) = \emptyset$. When both s and t are points, the claim is trivial. Otherwise, assume to the contrary that some $u \in I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(t)$.

We divide into the following cases: (i) $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$; (ii) $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$; (iii) $\vec{\alpha} \nvdash B \sqsubset s$ and $\vec{\alpha} \vdash B \sqsubset t$; (iv) $\vec{\alpha} \vdash B \sqsubset s$ and $\vec{\alpha} \nvdash B \sqsubset t$. (i) contradicts $\vec{\alpha} \vdash s \mapsto t$. For (ii), by the definitions of $I_{\vec{\alpha},B}(s)$ and $I_{\vec{\alpha},B}(t)$, we have $\vec{\alpha} \vdash u \sqsubset s$ and $\vec{\alpha} \vdash u \sqsubset t$, which contradict $\vec{\alpha} \vdash s \mapsto t$. For (iii), by the definition of $I_{\vec{\alpha},B}(s)$, we have $\vec{\alpha} \vdash u \sqsubset s$. By the definition of $I_{\vec{\alpha},B}(t)$, we have (iii-1) $\vec{\alpha} \vdash u \sqsubset t$ or (iii-2) $\vec{\alpha} \nvdash u \Box B$. (iii-1), together with $\vec{\alpha} \vdash u \sqsubset s$, contradicts $\vec{\alpha} \vdash s \mapsto t$. For (iii-2), $\vec{\alpha} \vdash u \sqsubset s$, $\vec{\alpha} \vdash s \mapsto t$, and $\vec{\alpha} \vdash B \sqsubset t$ imply, by Lemma 2.11(3), that $\vec{\alpha} \vdash u \mapsto B$, which contradicts $\vec{\alpha} \nvdash u \Box B$. (iv) is similar to (iii).

3. When all relations of an existential point x of $\mathsf{rel}(\vec{\alpha})$ are $x \sqsubset A_1, \dots, x \sqsubset A_i, x \mapsto B_1, \dots, x \mapsto B_j$, we show that there exists some $m \in M_{\vec{\alpha},B}$ such that $m \in I_{\vec{\alpha},B}(A_1) \cap \cdots \cap I_{\vec{\alpha},B}(A_i) \cap \overline{I_{\vec{\alpha},B}(B_1)} \cap \cdots \cap \overline{I_{\vec{\alpha},B}(B_j)}$.

For every $1 \leq k \leq i$, we have $\vec{\alpha} \vdash x \sqsubset A_k$ since $x \sqsubset A_k \in \mathsf{rel}(\vec{\alpha})$. Hence, we have $x \in I_{\vec{\alpha}}(A_k)$, that implies $x \in I_{\vec{\alpha},B}(A_k)$.

For every $1 \leq l \leq j$, in order to show $x \notin I_{\vec{\alpha},B}(B_l)$, we assume to the contrary that $x \in I_{\vec{\alpha},B}(B_l)$. Let $B_l \equiv B$ or $\vec{\alpha} \vdash B \sqsubset B_l$. Then, by the definition of $I_{\vec{\alpha},B}(B_l)$, we have (i) $\vec{\alpha} \vdash x \sqsubset B_l$ or (ii) $\vec{\alpha} \nvdash x \Box B$. (i) leads to contradiction since we have $\vec{\alpha} \vdash x \mapsto B_l$ by the fact $x \mapsto B_l \in \mathsf{rel}(\vec{\alpha})$. For (ii), by the fact $\vec{\alpha} \vdash x \mapsto B_l$ and $\vec{\alpha} \vdash B \sqsubset B_l$, we have $\vec{\alpha} \vdash x \mapsto B$, which contradicts to (ii). The other case ($B_l \not\equiv B$ and $\vec{\alpha} \nvdash B \sqsubset B_l$) is immediate.

Therefore, we have $x \in I_{\vec{\alpha},B}(A_1) \cap \cdots \cap I_{\vec{\alpha},B}(A_i) \cap \overline{I_{\vec{\alpha},B}(B_1)} \cap \cdots \cap \overline{I_{\vec{\alpha},B}(B_j)}$. ■

Using the two kinds of canonical models introduced so far, we prove the following *atomic completeness*, from which completeness (Theorem 3.10) of GDS is derived.

Proposition 3.9 (Atomic completeness). *Let $\mathcal{D}_1, \dots, \mathcal{D}_n$ be a set of diagrams which is consistent. Let β be a minimal diagram. If $\mathcal{D}_1, \dots, \mathcal{D}_n \models \beta$, then $\mathcal{D}_1, \dots, \mathcal{D}_n \vdash \beta$ in* GDS.

Proof. We first consider the case where the premise diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ are restricted to minimal diagrams $\alpha_1, \dots, \alpha_n$. Then we extend to the general case. We denote by $\vec{\alpha}$ the set of given minimal diagrams. Assume $\vec{\alpha} \models \beta$. When β is $s \bowtie t$, we immediately have $\vec{\alpha} \vdash s \bowtie t$ since it is an axiom. Otherwise, we divide into the following two cases according to the form of β.

(1) When β is of the form $s \sqsubset t$, by the assumption $\vec{\alpha} \models s \sqsubset t$, we have, in particular for the canonical model of Definition 3.4, $M_{\vec{\alpha}} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha}} \models s \sqsubset t$. Then, since $M_{\vec{\alpha}} \models \vec{\alpha}$ by Lemma 3.5, we have $M_{\vec{\alpha}} \models s \sqsubset t$, i.e., $I_{\vec{\alpha}}(s) \subseteq I_{\vec{\alpha}}(t)$. Since $s \in I_{\vec{\alpha}}(s)$ by Definition 3.4, we have $s \in I_{\vec{\alpha}}(t)$, that is, $\vec{\alpha} \vdash s \sqsubset t$ in GDS.

(2) When β is of the form $s \mathrel{\vdash\!\dashv} t$, observe that if s and t are both points, then the assertion is trivial since β is an axiom in that case. Otherwise, we assume, without loss of generality, that t is a named circle B. By the assumption $\vec{\alpha} \models s \mathrel{\vdash\!\dashv} B$, we have, in particular for the canonical model of Definition 3.7, $M_{\vec{\alpha},B} \models \vec{\alpha} \Rightarrow M_{\vec{\alpha},B} \models s \mathrel{\vdash\!\dashv} B$. Then since $M_{\vec{\alpha},B} \models \vec{\alpha}$ by Lemma 3.8, we have $M_{\vec{\alpha},B} \models s \mathrel{\vdash\!\dashv} B$, i.e., $I_{\vec{\alpha},B}(s) \cap I_{\vec{\alpha},B}(B) = \emptyset$. Hence we have $s \notin I_{\vec{\alpha},B}(B)$ and $B \notin I_{\vec{\alpha},B}(s)$. Then by the definition of $I_{\vec{\alpha},B}(B)$ and $I_{\vec{\alpha},B}(s)$ of Definition 3.7, we have $\vec{\alpha} \not\vdash s \sqsubset B$, and $\vec{\alpha} \not\vdash B \sqsubset s$ and $\vec{\alpha} \vdash s \Box B$ for some $\Box \in \{\sqsubset, \sqsupset, \mathrel{\vdash\!\dashv}\}$. Therefore, we have $\vec{\alpha} \vdash s \mathrel{\vdash\!\dashv} B$ in GDS.

Next, we extend the premises to general diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ instead of minimal diagrams $\vec{\alpha}$. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$.

Let $\mathsf{rel}(\vec{\mathcal{D}}) = \{R_1, \ldots, R_m\}$, then there is some $\vec{\alpha}$ such that $\mathsf{rel}(\vec{\mathcal{D}}) = \mathsf{rel}(\vec{\alpha})$. Hence, when $M \models \vec{\mathcal{D}}$, by the above argument, we have $\vec{\alpha} \vdash \beta$, i.e., there is a d-proof from $\alpha_1, \ldots, \alpha_m$ to β in GDS. Since each α_j is derived from some $\mathcal{D}_i$ by some applications of Deletion rule, we have $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \beta$. ■

By extending the conclusion diagram β of atomic completeness to a general (not restricted to minimal) diagram $\mathcal{E}$, we establish our completeness theorem.

Proposition 3.10 (Completeness)**.** *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be consistent. If $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, then $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E}$ in* GDS.

Proof (sketch). We define a canonical way to construct a d-proof of $\mathcal{E}$ from the given premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (see also Example 3.11 below):

(I) Minimal part: By using the atomic completeness theorem, we derive all of: (1) point-free minimal diagrams appearing in the conclusion; (2) pointed minimal diagrams consisting of every point contained in a premise; (3) pointed minimal diagrams obtained by combining the diagrams of (1) with an axiom A3 of the form $x_A \sqsubset A$ for a fresh point x_A and for a circle A contained in a premise or the conclusion.

(II) Venn part: By applying U1, U2 for the above pointed minimal diagrams of (I-2) and (I-3), we construct, for every point p of the Minimal part, a "Venn-like diagram", in which $A \bowtie B$ holds for any pair of circles in it, and which consists of p and all circles of the conclusion $\mathcal{E}$. If it is not possible to construct a Venn-like diagram for a point because of the constraint for determinacy, we do not construct it.

When no Venn-like diagram with a point is constructed, by applying U8 rule and the axiom A1, we construct a Venn-like diagram (without any point) which consists of all circles of $\mathcal{E}$.

(III) Modification part: By using U9, U10, and the point-free minimal diagrams obtained at the Minimal part (I-1), we modify forms and positions of circles of each Venn-like diagram above so that they correspond to those of the conclusion $\mathcal{E}$.

(IV) PI part: By applying PI, we unify all diagrams obtained at the Modification part.

(V) Renaming part: We finally obtain $\mathcal{E}$ by applying Ren, Del, and the copy operation described in Remark 2.8.

By the definition of our inference rules, a point in the conclusion comes, possibly with applications of Ren, from the following ways: (1) it is contained in a premise; (2) it is obtained by applying the copy operation of Remark 2.8; (3) it is provided by applying the axiom A3. Thus, at the Minimal part, we derive all possible pointed minimal diagrams consisting of these points. Also at the Renaming part, we obtain the conclusion by applying Ren, Del, and the copy operation.

At the Venn part, we are able to ignore a point for which we cannot construct a Venn-like diagram. This is because, for any point contained in the conclusion $\mathcal{E}$, the relations between the point and all circles of $\mathcal{E}$ are determined.

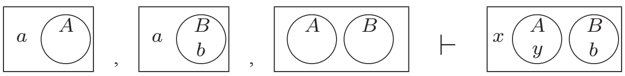
At the Modification part, we are able to apply U9, U10 rules undisturbedly. This is because, if diagrams $\mathcal{E}_1$ and $\mathcal{E}_2$ could not be unified because of the constraint for consistency of U9, U10 rules, since $\mathcal{E}_1$ and $\mathcal{E}_2$ are both derived from the premises, it means the premises are inconsistent.

At the PI part, we are able to unify all diagrams obtained so far. This is because all circles and points of those diagrams are essentially the same as those of the conclusion except for their names.

The correctness of the above canonical construction is shown in a similar way as (Mineshima et al., 2012). ■

Fig. 1 is an example of the canonical d-proof.

Example 3.11 (Canonical d-proof of GDS)**.** As an illustration of the canonical construction of d-proofs, let us consider the following diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{E}$:



$\mathcal{D}_1$ $\mathcal{D}_2$ $\mathcal{D}_3$ $\mathcal{E}$

We have a canonical d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ as in Fig. 1, where the derivation of $\mathcal{D}_{15}$ is omitted since it is similar to that of $\mathcal{D}_{13}$.

We first derive pointed minimal diagrams $\mathcal{D}_1, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6$ consisting of a point contained in a premise (I-2), and $\mathcal{D}_7, \mathcal{D}_8$ consisting of a fresh point comes from an axiom A3 (I-3). Next, at the Venn part, with U1 and U2 rules, we construct Venn-like diagrams $\mathcal{D}_9, \mathcal{D}_{10}$ and $\mathcal{D}_{11}$ each of which consists of the above point (a, b, z) and all circles A and B of $\mathcal{E}$. Then, at the Modification part, with U10 rule, we modify the above Venn-like diagrams so that positions of circles fit to those of the conclusion $\mathcal{E}$. Then, at the PI part, with PI rule, we unify $\mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{14}$, and $\mathcal{D}_{15}$ to obtain $\mathcal{D}_{18}$. Finally, by renaming the names of the points a and z, and by deleting w, we obtain the conclusion $\mathcal{E}$.

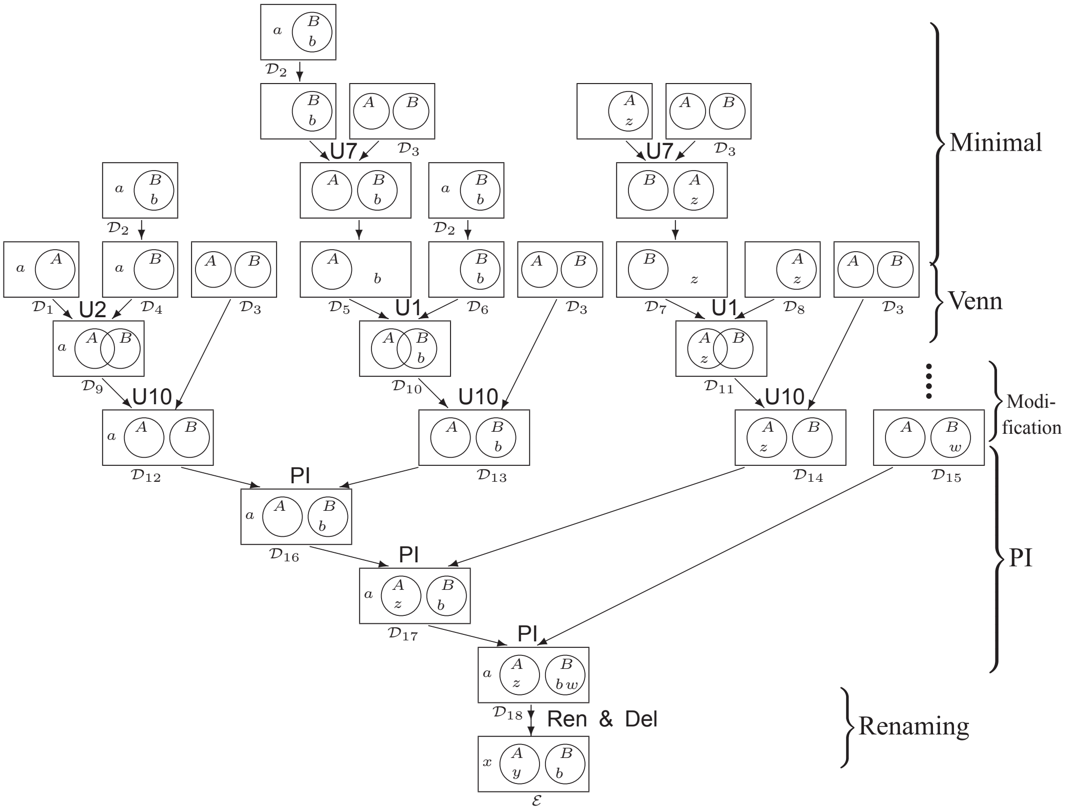


Fig. 1 Canonical d-proof

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A Unification rules of GDS

Unification rules are divided into three groups, Group (I), (II), and (III). The rules in Group (I) and (II) are classified according to the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α. The rule in Group (III) is the PI rule, where neither of two premise diagrams is restricted to be minimal.

(I) The case $\mathcal{D}$ and α share one object:

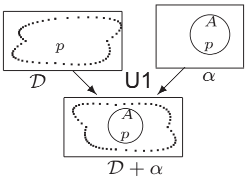
$\boxed{\text{U1}}$ Premises: $p \sqsubset A$ holds on α, and $p \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{p\}$.

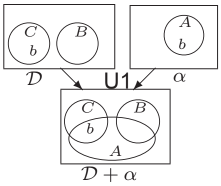
Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \sqsubset A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

The set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is specified as $\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$.

Schema of U1



Example of U1



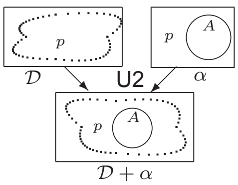
$\boxed{\text{U2}}$ Premises: $p \vdash\!\dashv A$ holds on α, and $p \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{p\}$.

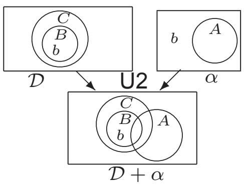
Operation: Add the circle A to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \vdash\!\dashv A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

Schema of U2



Example of U2

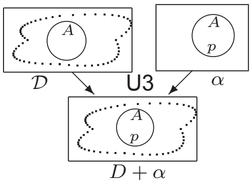


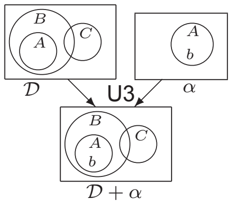
$\boxed{\text{U3}}$ Premises: $p \sqsubset A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $A \sqsubset X$ or $A \vdash\!\dashv X$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point p to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \sqsubset A$ holds; (2) $p \mathbin{\vdash\!\dashv} q$ holds for all points q such that $q \sqsubset A$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \ \cup\ \mathsf{rel}(\alpha)\ \cup \\ \{p \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup\ \{p \mathbin{\vdash\!\dashv} X \mid A \mathbin{\vdash\!\dashv} X \in \mathsf{rel}(\mathcal{D})\} \cup\ \{p \mathbin{\vdash\!\dashv} q \mid q \in pt(\mathcal{D})\}$$



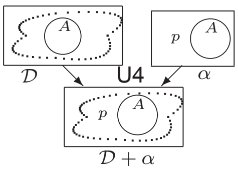


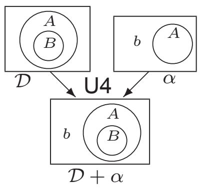
$\boxed{\text{U4}}$ **Premises:** $p \mathbin{\vdash\!\dashv} A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $X \sqsubset A$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point p to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \mathbin{\vdash\!\dashv} A$ holds; (2) $p \mathbin{\vdash\!\dashv} q$ holds for all points q such that $q \mathbin{\vdash\!\dashv} A$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{p \mathbin{\vdash\!\dashv} X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{p \mathbin{\vdash\!\dashv} q \mid q \in pt(\mathcal{D})\}$$



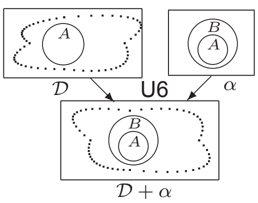


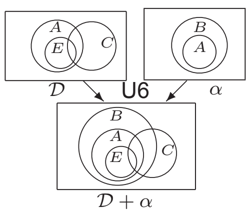
$\boxed{\text{U6}}$ **Premises:** $A \sqsubset B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $p \sqsubset A$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle B to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $A \sqsubset B$ holds; (2) $B \bowtie X$ holds for all circles $X (\not\equiv A)$ such that $A \sqsubset X$ or $A \mathbin{\vdash\!\dashv} X$ or $A \bowtie X$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \mathbin{\vdash\!\dashv} X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\} \\ \cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{p \sqsubset B \mid p \in pt(\mathcal{D})\}$$



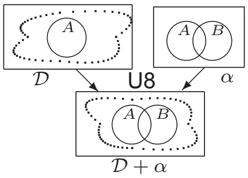


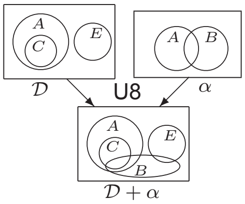
$\boxed{\text{U8}}$ **Premises:** $A \bowtie B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \emptyset$.

Operation: Add the circle B to $\mathcal{D}$ so that $B \bowtie X$ holds for all circles X of $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{B \bowtie X \mid X \in cr(\mathcal{D})\}$$





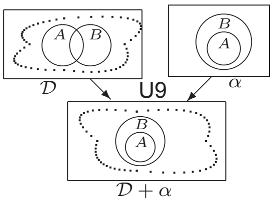
(II) The case $\mathcal{D}$ and α share two circles:

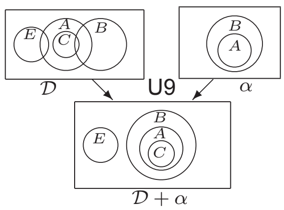
$\boxed{\text{U9}}$ **Premises:** $A \sqsubset B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \vdash\!\dashv B$ hold on $\mathcal{D}$.

Operation: Modify all circles X (including A) of $\mathcal{D}$ such that $X \sqsubset A$ holds so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $X \sqsubset B$ holds; (2) $X \Box t$ holds with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$ for all object t of $\mathcal{D}$ such that $t \sqsubset A, X \Box t \in \mathsf{rel}(\mathcal{D})$.

$$\begin{aligned}\mathsf{rel}(\mathcal{D}+\alpha) = \big(&\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\}\big) \\ &\cup \{X \sqsubset Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$



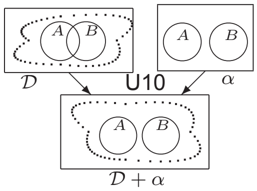


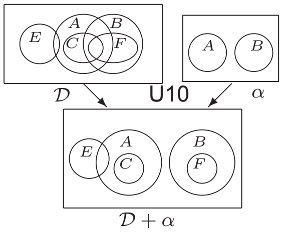
$\boxed{\text{U10}}$ **Premises:** $A \vdash\!\dashv B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$.

Operation: Modify all circles X (including A) and Y (including B) of $\mathcal{D}$ such that $X \sqsubset A$ and $Y \sqsubset B$, respectively hold on $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $X \vdash\!\dashv B$ holds; (2) $X \Box t$ holds with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$ for all object t of $\mathcal{D}$ such that $t \sqsubset A, X \Box t \in \mathsf{rel}(\mathcal{D})$; (3) $Y \vdash\!\dashv A$ holds; (4) $Y \Box s$ holds with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$ for all object s of $\mathcal{D}$ such that $s \sqsubset B, Y \Box s \in \mathsf{rel}(\mathcal{D})$.

$$\begin{aligned}\mathsf{rel}(\mathcal{D}+\alpha) = \big(&\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big) \\ &\cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$



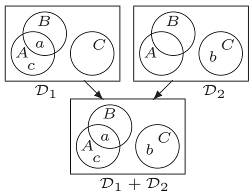


(III) Neither of two premise diagrams is restricted to be minimal:

$\boxed{\text{PI (Point Insertion)}}$ Premises: $X\Box Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X\Box Y \in \mathsf{rel}(\mathcal{D}_2)$ holds for any circles X, Y with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, and $pt(\mathcal{D}_2) = \{p\}$ such that $p \notin pt(\mathcal{D}_1)$.

Operation: Add the point p to $\mathcal{D}_1$ so that the following conditions are satisfied on $\mathcal{D}_1 + \mathcal{D}_2$: (1) $p\Box t$ of $\mathsf{rel}(\mathcal{D}_2)$ holds for all objects t; (2) $p \vdash\!\dashv q$ holds for all $q \in pt(\mathcal{D}_1)$.

$\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{p \vdash\!\dashv q \mid q \in pt(\mathcal{D}_1)\}$



$\boxed{\text{Del (Deletion)}}$ Premise: $\mathcal{D}$ contains an object s.

Constraint: $\mathcal{D}$ is not minimal.

Operation: Delete the object s from $\mathcal{D}$.

$\mathsf{rel}(\mathcal{D} - s) = \mathsf{rel}(\mathcal{D}) \setminus \{s\Box t \mid t \in ob(\mathcal{D}), \Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}\}$

要旨

健全性定理や完全性定理に代表される論理学的分析はこれまで、言語（記号）表現に基づく推論体系に対して行われてきたが、近年になって図形的な表現に基づく推論体系の論理学的分析が注目されている。とくにオイラー図は18世紀にLeonhard Eulerによって導入されて以来、論理学や数学などさまざまな分野でインフォーマルに用いられてきたが、1990年代になってから、現代論理学的手法による研究がなされるようになった。Mineshima, Okada , & Takemura (2012)のオイラー図推論体系では、オイラー図は円と点からなる平面図形として形式的に定義され、与えられたオイラー図どうしを合成する操作が推論規則として形式化されている。本稿では、Mineshima, Okada, & Takemura (2012)のオイラー図推論体系に、存在量化子に対応するものを加えて拡張する。さらに、その拡張されたオイラー図推論体系の完全性定理を証明する。